Stiefel-Whitney Classes and Persistent Stiefel-Whitney Classes

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Outline

Introduction

Topological Invariants
- Homology
- Cohomology
- Betti Numbers

Stiefel-Whitney Classes

Persistent theory

Persitent Stiefel-Whitney Classes

Summary

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Motivation

Generalise the theory of persistent homology, a powerful tool for studying the shape of data.

Introduction

Topology is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing.

In topology, the most basic equivalence is a homeomorphism, which allows spaces that appear quite different in most other subjects to be declared equivalent in topology.

Introduction

Homology, cohomology and Betti numbers are some invariants that can distinguish spaces up to some condition.

Some natural cohomology classes called characteristic classes, associated to vector bundles on spaces, measure in some way how a vector bundle is twisted, or nontrivial.

There are four main characteristic classes:

1. Stiefel-Whitney classes
2. Chern classes
3. Pontryagin classes
4. The Euler class

Introduction

Persistent homology (a powerful tool used in Topological Data Analysis) has found its applications in many fields, including, image processing, medicine, network analysis, material science, sensor network, biology, and fields in mathematics such as dynamical systems. [Kerber (2016), Tillmann(2018)]

A natural extension of homology is cohomology.

Topological Invariants

It is a popular saying that to a topologist, cannot distinguish a coffee mug from a doughnut.

The surfaces of a donut and a coffee cup (with one handle) are considered equivalent because both have a single hole.

An invariant is a property of an object that remains unchanged upon transformations such as scaling or rotations.

Examples
1. Dimension: $R^1 \neq R^2$ since $1 \neq 2$. A circle $\neq$ sphere
2. Determinant: If matrices A and B are similar, their determinants are equal.

In general,for topological spaces, an invariant informs us whether the spaces are different or same with respect to the invariant.

Examples of Invariants

Examples of invariants that classify topological spaces up to homeomorphism:

- Homology
- Cohomology
- Betti number
- Euler Characteristic

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Homology

Homology - a notion in Algebraic Topology associating a sequence of abelian groups or modules with a topological space.

It can be very difficult to compute the homology of arbitrary topological spaces.

Spaces are approximated by combinatorial structures called *simplicial complexes* for which homology can be easily computed algorithmically.

Simplicial Complexes

The $(n+1)$-tuple $(x_0, x_1, ..., x_n)$ (in $\mathbb{R}^d$)is said to be *affinely independent* if the set of vectors $\{x_j - x_0 | j = 1, 2, ..., n\}$ is a linearly independent set.

A set of $k+1$ affinely independent points in $\mathbb{R}^d$ is a k*-simplex* denoted σ which can be represented by $[v_1; \cdots ; v_k]$ and each v_i is called a *vertex* of the simplex.

Simplicial Complexes

Remark

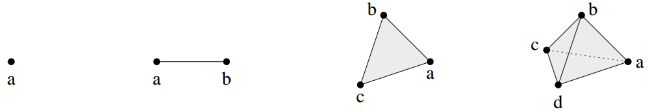
0-simplex is a point or a vertex or a node

1-simplex is a connection of a pair of nodes forming an edge

2-simplex consists of three nodes and three edges, with a face, forming a triangle

3-simplex consists of four nodes, six edges and four triangles, with four faces, forming a tetrahedron, and so on.

Simplices



vertex {a}

edge {a, b}

triangle {a, b, c}

tetrahedron {a, b, c, d}

Simplices

Definition 1

A simplex σ is a *face* of τ if the vertices of σ form a subset of the vertices of τ and is denoted $\sigma \leq \tau$

Remark

- A 0-simplex has only one face, a 0-face $\{a\}$.
- A 1-simplex has two 0-faces $\{a\}, \{b\}$ and a 1-face $\{a, b\}$.
- A 2-simplex has three 0-faces $\{a\}, \{b\}, \{c\}$, three 1-faces $\{a, b\}, \{a, c\}, \{b, c\})$ and one 2-face $\{a, b, c\}$.

Simplicial Complex

A *simplicial complex* is a set of simplices which are convex hulls of affinely independent points.

The collection of all k-simplices in X is denoted X^k.

The *dimension* of a simplicial complex is the highest dimension of its simplices.

The *vertices* are the 0-simplices.

A *simplicial complex* is obtained by gluing together finitely many simplices.

Simplicial Chains

Definition 2

For X a simplicial complex, a *simplicial k-chain* on X is a finite formal sum of all simplices X^k,

$$c = \sum_i a_i \sigma_i$$

such that $a_i \in \mathbb{Z}_p$, p a prime number.

Simplicial Chains

Remark

- Simplicial 0-chains are formal sums of 0-simplices- the vertices
- Simplicial 1-chains are formal sums of 1-simplices;
- Simplicial 2-chains are formal sums of 2-simplices

Chain Groups

Definition 3

The set of all simplicial k-chains in X with the addition given by the addition of coefficients forms a finitely generated abelian group called the k*th chain group* denoted $C_k(X)$.

The concept of chain groups lead to algebraic expression of *boundary*

Boundary of a k- Simplex

Definition 4

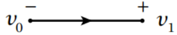
The *boundary operator* $\partial_k : C_k(X) \to C_{k-1}(X)$ is a linear mapping that maps a k-simplex to a $k-1$ simplex

$$\partial_k([v_0, \cdots, v_k]) = \sum_{i=0}^{k} (-1)^i ([v_0, \cdots, \hat{v}_i, \cdots, v_k])$$

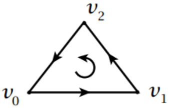
where $\hat{v}_i$ denotes removal of v_i.

A k- chain is a *boundary* if it is in the image of ∂_{k+1}

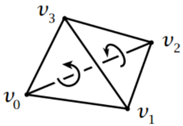
Boundary of an edge, triangle and tetrahedron



$$\partial[v_0, v_1] = [v_1] - [v_0]$$



$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$



$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

Homology Group

Definition 5

The k*th homology group* is the quotient group

$$H_k(X, \mathbb{Z}_p) = Ker(\partial_k)/Im(\partial_{k+1})$$

containing equivalence classes of k-cycles.

Since,

$$\partial_k \circ \partial_{k+1} = 0$$

$Im(\partial_{k+1})$ is a subgroup of $Ker(\partial_k)$
That is,
every boundary is also a cycle.

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Cohomology

Cohomology is a sequence of abelian groups associated to topological spaces and is defined from a cochain complex.

Definition 6

A *cochain complex* is a function on the chain group of the homology theory.
That is,
A k-*cochain complex* is a function $\alpha : X^k \to R$, R is a commutative ring.
The set of all k-cochains form *cochain group*, denoted by $C^k(X, R)$.
The *coboundary operator* $d_k : C^k(X, R) \to C^{k+1}(X, R)$ maps a cochain to a cochain one dimension higher.

$$d_k(\alpha)([v_0, \cdots, v_k]) = \sum_{i=0}^{k} (-1)^i \alpha([v_0, \cdots, \hat{v}_i, \cdots, v_k]),$$

for a k-cochain α.

Remark

Note that the two operators, d_k and ∂_{k+1} are transposes to each other.

Cohomology Group

Definition 7

A k-cochain is called a *coboundary* if it is in the image of d_{k-1}.

A k-cochain is called a *cocycle* if it is image under d_k is 0.

The coboundary operators have the property that $d_k \circ d_{k-1} = 0$.

The kth *cohomology group* is the quotient group

$$H^k(X, R) = Ker(d_k)/Im(d_{k-1})$$

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Betti Number

Definition 8

The kth Betti number $\beta_k(X)$ of a space X is the rank of the kth homology group of X, $H_k(X)$.
That is,

$$\beta_k(X) = rank(H_k(X))$$

Remark

kth Betti number distinguishes topological spaces by counting the number of kth dimensional holes in the space.

Equivalently, β_k is the maximum number of k-dimensional curves that can be removed while the object remains connected.

- β_0 is the number of connected components
- β_1 is the number of tunnels or circular holes
- β_2 is the number of voids or cavities

Betti Numbers of Some Spaces

Example 9

- The Betti number sequence for a point is $\{1, 0, 0, 0, 0, 0\}$
- The Betti number sequence for a circle is $\{1, 1, 0, 0, 0, \cdots\}$
- The Betti number sequence for a sphere is $\{1, 0, 1, 0, 0, \cdots\}$
- The Betti number sequence for a torus is $\{1, 2, 1, 0, 0, \cdots\}$
- The Betti number sequence for a Projective space is $\{1, 0, 0, 0, 0, \cdots\}$

Characteristic Classes

Characteristic classes are cohomology classes which are associated to the base spaces of vector bundles and encodes information of such bundles.

They measure how a vector bundle is twisted or non-trivial.

There are four main types:
1. Stiefel-Whitney classes
2. Chern classes
3. Pontryagin classes
4. The Euler class

Stiefel-Whitney Classes

Stiefel-Whitney classes were first constructed by H. Whitney and E. Stiefel independently in 1935.

Since then there have been many other constructions of Stiefel-Whitney classes.

There are many approaches to Stiefel-Whitney (SW) classes but we give the axiomatic approach

Stiefel-Whitney Classes

Let $\xi = (E, p, B)$ be a real n - dimensional vector bundle over a space B.

There is a class $w(\xi) \in \ H^*(B, \mathbb{Z}_2)$ with the following axioms:

I $w(\xi) = w_0(\xi) + w_1(\xi) + \cdots + w_n(\xi)$ *where* $w_i(\xi) \in H^i(B, \mathbb{Z}_2)$ $i = 0, 1, 2, \ldots,$ $w_i(\xi) = 0$ for $i > n$ *and* $w_0(\xi) = 1 \in \ H^0(B, \mathbb{Z}_2)$.

II **Naturality** *If* $\xi = (E, p, B)$ *is a real vector bundle and* $f : B_1 \to B$ *is a map then*

$$w_i(f^*\xi) = f^*w_i(\xi)$$

III **The Whitney Product formula** *If* ξ *and* η *are real vector bundles over the same base space then*

$$w(\xi \oplus \eta) = w(\xi)w(\eta)$$

IV For the canonical line bundle γ_1 over $\mathbb{R}P^\infty$ [the element $w_1(\gamma_1)$ is the generator of the polynomial ring $H^*(\mathbb{R}P^\infty, \mathbb{Z}_2)$.]

$$w_1(\gamma_1) \neq 0.$$

Stiefel-Whitney Classes

$$w(\xi) = w_0(\xi) + w_1(\xi) + w_2(\xi) + \cdots$$

is *the total Stiefel-Whitney class of* ξ

and

$w_i(\xi)$ *the* i*-th Stiefel-Whitney class of* ξ.

Remark

The proof of existence of the cohomology classes and that the axioms uniquely characterize cohomology classes satisfying the axioms of Stiefel-Whitney classes can be found in Milnor and Stasheff (1974) and Husemoller (1994)

Stiefel-Whitney Classes

Remark

The Stiefel-Whitney class of a manifold M, denoted $w(M)$ is the Stiefel-Whitney class $w(\tau(M))$ of its tangent bundle.

Properties of Stiefel-Whitney Classes

These properties follow trivially from the definition

- If ε^n is a trivial n-plane bundle then $w(\varepsilon^n) = 1$.
  Proof. Let $\star$ be a point space, then since every trivial bundle is isomorphic to the induced bundle of $\star$ by a map, we have

$$\begin{aligned} w_i(\epsilon^n) &= w_i(f^*(\star)) \\ &= f^* w_i(\star) \\ &= 0 \ \text{ if } i > 0 \qquad \text{by naturality axiom} \end{aligned}$$

  since $w_i(\star) \in H^i(\star) = 0$ for $i > 0$.
- For any n-plane bundle η and any line bundle ξ the total Stiefel-Whitney class of their tensor product is given by:

$$\begin{aligned} w(\eta \otimes \xi) &= (1 + w_1(\xi))^n + w_1(\eta)(1 + w_1(\xi))^{n-1} \\ &\quad + w_2(\eta)(1 + w_1(\xi))^{n-2} + \cdots + \\ &\quad + w_{n-1}(\eta)(1 + w_1(\xi)) + w_n(\eta). \end{aligned}$$

  This can be shown by induction on n the dimension of η.

Stiefel-Whitney Classes of the Sphere and Projective Spaces

$$w(S^n) = 1$$

$$w(\mathbb{R}P^{n-1}) = (1+a)^n = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) a^i$$

$$w_k(\mathbb{R}P^{n-1}) = \prod_{i>0} \left(\begin{array}{c} n_i \\ \\ k_i \end{array} \right) \ \mathrm{mod} 2 \ w_1^k$$

Moreover, if n is even $w_k(\mathbb{R}P^{n-1})$ vanishes for any odd k.

Properties of Stiefel-Whitney Classes

If τ is the tangent bundle of a manifold M, then M is orientable if and only if $w_1(\tau) = 0$.

M admits a spin structure if and only if $w_2(\tau) = 0$

Applications of Stiefel-Whitney Classes

Among other applications, we used Stiefel-Whitney classes to

1. evaluate the span of manifolds [Ajayi and Ilori (2002), Ajayi(2014),]
2. estimate non-embedding /non-immersion results [Ajayi and Ilori (1999), Ilori and Ajayi (2008), Ajayi (2010)]
3. determine the Lusternik - Shnirelmann category Cat(X) of a space X. [Ilori and Ajayi (2000)]

Persistence

Persistence is a measure of topological attributes.

It is called persistence, as it ranks attributes by their life time in a filtration - their persistence in being a feature in the face of growth.

The main premise of persistence is that a significant topological attribute must have a long life-time in a filtration:

The attribute persists in being a feature of the growing complex.

Persistent Homology

Persistent homology - a fundamental tool in Topological Data Analysis.

Homology defines and counts holes; persistent homology measures holes

Studies the evolution of k-dimensional holes along a sequence of simplicial complexes (that is, a filtration).

The set of intervals representing birth and death times of k-dimensional holes along a filtration is the *persistence barcode*

k-dimensional holes with short lifetimes are *topological noise*,

those with a long lifetime are *topological feature* associated to the given data.

Persistent Homology

Simplicial homology is homology on a single simplicial complex.

Building on this, persistent homology is homology on a growing sequence of simplicial complexes called filtration.

For a simplicial complex K, a *filtration* is a sequence of sets

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^{\max} = K$$

such that each K^i is a simplicial subcomplex of K.

Persistent Homology

The definitions that lead to the construction of persistent homology are analogous to those in simplicial homology.

The definitions are applied to each simplicial complex in a filtration, using superscripts to denote the index in the filtration.

Thus, in a filtration, the ith simplicial complex K^i gives rise to its own kth chain groups $(K_k^i, +)$, each of which consists of the set of simplicial k-chains, together with formal addition.

The sequence of these kth chain groups K_k^i together with boundary maps

$$\partial_k^i \longrightarrow K_{k-1}^i$$

forms the chain complex $(K_\bullet^i, \partial_\bullet^i)$.

Persistent Homology

Thus, as we have simplicial homology,

1. the kth chain group, K_i^k, which is a finitely generated free abelian group consisting of the set of simplicial k-chains together with formal addition;
2. the kth cycle module, Z_k^i, which is the kernel of the boundary map ∂_k^i, that is $ker\partial_k^i$
3. the kth boundary module, B_k^i, which is the image of the boundary map ∂_k^i, that is $im\partial_k^i$; and
4. the kth homology module, H_k^i, which is given as

$$H_k^i = Z_k^i / B_k^i$$

Persistent Module

Persistent module is defined as follows:

Definition 10

For a positive integer p, the p-persistent kth homology module of the ith simplicial complex K_i is defined as,

$$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

The p-persistent kth Betti number of the ith simplicial complex K^i , denoted by β_k^{i+p}, is the rank of $H_k^{i,p}$

Birth and Death of Topological Features

Definition 11

Let K be a simplicial complex and let

$$\emptyset = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^{\max} = K$$

be a *filtration* on K.

A topological feature in a given dimension is said to be *born* at step m of the filtration if the homology module of K^m is the first homology module of the simplicial complex in the filtration to include the feature.

The topological feature *dies* at filtration step n if it is present in the homology module of K^{n-1} but not in the homology module of K^n

Other Variants / Extensions of Persistence

Although Persistence was originally defined for homology, recent research has developed persistent approaches to other algebraic topology invariants including Persistent Cohomology, Persistent Characteristic Classes, Persistent Steenrod Squares.

Cohomology has been a valuable approach in persistence which has been found to accelerate computations significantly and more powerful than homology because of its ring structure.

Persistent Stiefel-Whitney Classes

For the definition of Persistent Stiefel-Whitney classes, we follow [Tinarrage(2020)].
We define a vector bundle filtration $(X^t)_{t\geq 0}$, which is the Čech filtration of X and maps $\xi^t : X^t \longrightarrow G_d(\mathbb{R}^m)$

1. By embedding $G_d(\mathbb{R}^m) \hookrightarrow M((R)^m)$, then $\tilde{X} := \{(x, \xi(x)), x \in X\}$ is a subset of $\mathbb{R}^n \times M(\mathbb{R}^m)$
2. Let $(\tilde{X}^t)_{t\geq 0}$ be the Čech filtration of $\tilde{X}$ in the natural space $\mathbb{R}^m \times M(\mathbb{R}^m)$, endowed with the metric $\|k(x, A)\| = \sqrt{\|x\|_2^2 + \|A\|_F^2 q}$
3. We can define extended maps ξ^t as follows:

$$\xi^t : \tilde{X}^t \longrightarrow G_d(\mathbb{R}^m)$$

$$(x, A) \mapsto \mathrm{proj}(A, G_d(\mathbb{R}^m))$$

Persistent Stiefel-Whitney Classes

The data of $(\tilde{X}^t)_{t\geq 0}$ and $(\xi^t : \tilde{X}^t \longrightarrow G_d(\mathbb{R}^m)_{t\geq 0})$ is the Čech bundle filtration of $\tilde{X}$.

For every $t \geq 0$, we have the ith Stiefel-Whitney class of $(\tilde{X}^t, \xi^t)$

$$w_i(\xi^t) = (\xi^t)^*(w_i)$$

where $(\xi^t)^* : H^*(\tilde{X}^t) \longleftarrow H^* G_d(\mathbb{R}^m)$

Definition 12

The ith *persistent Stiefel-Whitney class* of $\tilde{X}$ is the collection $(w_i(\xi^t))_{t\geq 0}$

There exists a maximal value $t^{\max}$ such that for all $t \in [0, t^{\max})$, the maps ξ^t are well-defined.

Definition 13

The persistent Stiefel-Whitney class $(w_i(\xi^t))_t$ is defined for every $t \in [0, t^{\max})$

Summary

- Brief introduction into theory of homology, cohomology and characteristic classes
- Reviewed persistent theory on homology and Stiefel-Whitney
- Look forward to finding possible applications of persistent Stiefel-Whitney classes in topological data analysis

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Thank you