☞ Symmetry Analysis of Differential Equations and their Applications: Recent Case Studies

Presentation by
S. Moyo

Durban University of Technology
Office of the DVC Research, Innovation & Engagement, Milena Court

Co-authors: M.O. Aibinu and T. Mkhize
Durban University of Technology

20 April 2022

Outline

☞ Differential equations

In recent years (See e.g, [1]),

- many researchers are interested in the the study of Differential Equations (DEs).
- using DEs is emerging as an area of substantial activity across many applications.
- there exists inquisition on well-defined method to systematically solve fractional differential equations.
- developing numerical or analytical solutions for these fractional mathematical models are crucial issues.

☞ Solution of partial differential equations

- There exists no well-defined method to systematically solve Fractional Partial Differential Equations (FPDEs).
- To deal with FPDEs, efforts have been made to develop techniques such as (See e.g, [2])
    - sine-cosine method;
    - variational iteration method;
    - homotopy perturbation method;
    - Jacobi elliptic function method;
    - first integral method;
    - tanh function and extended tanh function methods;
    - simplest equation method;
    - exponential function method;
    - Weierstrass elliptic function expansion method;
    - G'/G-expansion method.

☞ Symmetry analysis

The symmetry analysis as applied to differential equations

- transforms solutions into solutions while leaving the set of all solutions invariant.
- can be used to obtain new solutions from known ones.
- can be used to classify equations into equivalence classes according to their symmetry groups.
- can be used to obtain exact analytic solutions that are invariant under some group invariant solutions.

Lie group analysis is one of the systematic symmetry techniques

- which applies Lie group theory to differential equations.
- which has received considerable attention in the literature (See e.g, [3, 4])
- is a most powerful tool which gives a systematic way of determining transformations that can:
  - map equations from nonlinear form to linear forms;
  - transform systems of equations to a set that is more amenable to analyze.

☞ Peculiarities of this present study

This presentation will focus on

- unveiling the Lie group analysis method in an explicit way;
- demonstration of the applications of Lie group analysis method to to systematically solve FPDEs;
- illustration of Lie group analysis method with selected recent case studies.

☞Example 1 on using variational iteration method

Consider the nonlinear second-order pantograph equation

$$\begin{aligned} y''(x) &- 8y\left(\frac{x}{2}\right) + xy'(x) = 2, \\ y(0) &= 0, \\ y'(0) &= 0. \end{aligned} \tag{2.1}$$

Solution: Taking the Sumudu Transform (ST) of (2.1) gives (See e.g., [5])

$$\frac{Y(u)}{u^2} - \frac{y(0)}{u^2} - \frac{y'(0)}{u} = S\left[8y\left(\frac{x}{2}\right) - xy'(x) + 2\right]. \tag{2.2}$$

Since $y(0) = 0$ and $y'(0) = 0$, equation (2.2) gives

$$\frac{Y(u)}{u^2} = S\left[8y\left(\frac{x}{2}\right) - xy'(x) + 2\right]. \tag{2.3}$$

☞ Example 1 (Continues)

The variational iteration formula is given by

$$Y_{n+1}(u) = Y_n(u) + \\ + \quad \varphi(u)\left(\frac{Y_n(u)}{u^2} - S\left[8y_n\left(\frac{x}{2}\right) - xy_n'(x) + 2\right]\right). \quad (2.4)$$

The Lagrange multiplier is obtained as

$$\varphi(u) = -u^2. \quad (2.5)$$

Taking the inverse-Sumudu transform gives the exact solution after second iteration as $y(x) = x^2$.

☞ Example 2 on using variational iteration method

Consider the first order nonlinear delay differential equation of pantograph type

$$y'(x) - 2xy^4\left(\frac{x}{2}\right) = 0, \; y(0) = 1. \quad (2.6)$$

- Take the ST of (2.6).
- Generate the variational iteration formula.
- Obtain the Lagrange multiplier.

☞ Iterations for Example 2

$$y_1(x) = S^{-1}\left[-u(\frac{-1}{u})\right] = S^{-1}[1] = 1.$$

$$\begin{aligned} y_2(x) &= y_1(x) + S^{-1}\left[u\left(S\left[2xy_1^4\left(\frac{x}{2}\right)\right]\right)\right] \\ &= 1 + x^2. \end{aligned}$$

$$\begin{aligned} y_3(x) &= y_1(x) + S^{-1}\left[u\left(S\left[2xy_2^4\left(\frac{x}{2}\right)\right]\right)\right] \\ &= 1 + x^2 + \frac{x^4}{2}. \end{aligned}$$

☞ Iterations for Example 2

$$
\begin{aligned}
y_4(x) &= y_1(x) + S^{-1}\left[u\left(S\left[2xy_3^4\left(\frac{x}{2}\right)\right]\right)\right] \\
&= 1 + \frac{1}{1!}x^2 + \frac{1}{2!}\left(x^2\right)^2 + \frac{1}{3!}\left(x^2\right)^3 \\
&= \sum_{i=0}^{3} \frac{1}{i!}\left(x^2\right)^i .
\end{aligned}
$$

$$
y_n(x) = \sum_{i=0}^{n} \frac{1}{i!}\left(x^2\right)^i, n \in \mathbb{N},
$$

which tends to e^{x^2} as $n \to \infty$.

- The exact solution is known to be $y(x) = e^{x^2}$.

☞ Fractional derivative

Given a Riemann-Liouville (RL) fractional integral operator of order $\alpha \in [n-1, n),\ n \in \mathbb{N}$, α derivative of $f(t)$ is given as:

- RL fractional derivative:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, & 0 < \alpha < 1, \\ (f^n(t))^{\alpha-n}. \end{cases}$$

- Caputo fractional derivative:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau, 0 < \alpha < 1.$$

- Here are some of the properties of RL derivative.

(i) $D_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \gamma > 0.$

(ii) $D_t^{\alpha}\left[f(t)g(t)\right] = \sum_{n=0}^{\infty} \binom{\alpha}{k} D_t^n f(t) D_t^{n-\alpha} g(t)$, where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1-n)}$.

(iii) $D_t^{\alpha}\left[f\left(g(t)\right)\right] = \sum_{n=0}^{\infty} \frac{U_n}{n!} \frac{d^n f(z)}{dz^n}|_{z=g(t)}$, where

$$U_n = \sum_{n=0}^{\infty} (-1)^k \binom{n}{k} g^k(t) \partial_t^{\alpha} \left(g^{n-k}(t)\right).$$

☞ Time-fractional generalized Burgers equation

Consider the time-fractional generalized Burgers equation [6]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = uu_x + u_{xx}, \tag{3.1}$$

where in the sense of RL fractional derivative,

- $0 < \alpha < 1$, x is space coordinate, t is time,
- $u(x,t)$ is unknown function, and
- $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the time-fractional derivative of the function $u(x,t)$.

☞ Exact solution of time-fractional generalized Burgers equation

- To apply the Lie group analysis method for the construction of exact solution of

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u u_x + u_{xx}, \qquad (3.2)$$

- the following function is introduced

$$U(\eta) = A\eta^k,$$

- where $\eta = xt^{-a}, \; a = \frac{\alpha}{2}$,
- parameters A and k are arbitrary real constants [1].

☞ Computation

- Follow the outline in the work of Costa et al. [7] to compute

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\beta} s^{-a} U\left(xs^{-a}\right) ds. \quad (3.3)$$

- Consider the case $\tau = \frac{s}{t}$, then (3.3) become

$$\begin{aligned}\frac{\partial^\beta u}{\partial t^\beta} &= \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t (1-\tau)^{-\beta} t^{1-a-\beta}\tau^{-a} U\left(\eta\tau^{-a}\right) d\tau \\ &= \frac{\partial}{\partial t}\left[t^{1-a-\beta}\left(F_\beta^{-a,a}U\right)(\eta)\right], \quad (3.4)\end{aligned}$$

where $\left(F_\beta^{-a,a}U\right)(\eta) = \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t (1-\tau)^{-\beta}\tau^{-a} U\left(\eta\tau^{-a}\right) d\tau.$

☞ Computation

- Recall that from $\eta = xt^{-a}$,

$$\frac{\partial}{\partial t} = \frac{d}{d\eta}\frac{\partial \eta}{\partial t} = -a\eta t^{-1}\frac{d}{d\eta}.$$

- Therefore we have that

$$\frac{\partial^\beta u}{\partial t^\beta} = t^{-a-\beta}\left[\left(1-a-\beta-a\eta\frac{d}{d\eta}\right)\left(F_\beta^{-a,a}U\right)(\eta)\right] \tag{3.5}$$

- Substitute (3.5) into

$$\frac{\partial^\alpha u}{\partial t^\alpha} = uu_x + u_{xx}, \tag{3.6}$$

to obtain

$$\left[\left(1-a-\beta-a\eta\frac{d}{d\eta}\right)\left(F_\beta^{-a,a}U\right)(\eta)\right] = UU_\eta + U_{\eta\eta}.$$

☞ Computation

- Follow the outline in the work of Costa et al. [7] to calculate the operator $\left(F_{\beta}^{-a,a}U\right)(\eta^k)$to obtain

$$\begin{aligned}\left(F_{\beta}^{-a,a}U\right)(\eta^k) &= \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t (1-\tau)^{-\beta}\eta^k\tau^{-a(1+k)}d\tau \\ &= \frac{\Gamma(1-a(1+k))}{\Gamma(2-a(1+k)-\beta)}A\eta^k. \end{aligned} \tag{3.7}$$

- Setting $\beta=\alpha,\ a=\frac{\alpha}{2}$ and using (3.7) makes $\left[\left(1-a-\beta-a\eta\frac{d}{d\eta}\right)\left(F_{\beta}^{-a,a}U\right)(\eta)\right]=UU_{\eta}+U_{\eta\eta}.$

to become

$$\frac{\Gamma(1+\alpha)}{\Gamma(2-2\alpha)}A\eta^k = A^2k\eta^{2k-1}+Ak(k-1)\eta^{k-2}. \tag{3.8}$$

☞ Exact solution

- To have the exact solution,

$$\frac{\Gamma(1+\alpha)}{\Gamma(2-2\alpha)} A\eta^k = A^2 k\eta^{2k-1} + Ak(k-1)\eta^{k-2}, \quad (3.9)$$

- must be invariant with respect to the similarity variable η.
- When $k = 1$, it is obtained that

$$A = \frac{\Gamma(1+\alpha)}{\Gamma(2-2\alpha)}, \quad (3.10)$$

such that the exact solution of

$$\frac{\partial^\alpha u}{\partial t^\alpha} = uu_x + u_{xx}, \quad (3.11)$$

is given as

$$u(x,t) = \frac{\Gamma(1+\alpha)}{\Gamma(2-2\alpha)} \frac{x}{t^{\frac{a}{2}}}. \quad (3.12)$$

☞ Exact solution of noninteger order derivatives

- Using Lie group analysis method, exact solution of time-fractional generalized Burgers equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u u_x + u_{xx}, \tag{3.13}$$

- is obtained as

$$u(x,t) = \frac{\Gamma(1+\alpha)}{\Gamma(2-2\alpha)} \frac{x}{t^{\frac{\alpha}{2}}}. \tag{3.14}$$

- Observe that the obtained exact solution
  - is valid for $t > 0$ and $x \in [0, \infty)$;
  - is called a very singular solution or dipole solution;
  - shows clearly that a change in noninteger order derivative value will affect the solution behavior in a fundamental way;
  - suggests that the noninteger order derivative can be used

References

B. Bira, T. R. Sekhar, D. Zeidan, (2018). *Math Meth Appl Sci.* 41:67176725.

I.B. Giresunlu, Y.S. *Ö*zkan, E. Yaşar, (2017). *Math. Meth. Appl. Sci.* 40, 39273936.

N.Kh. Ibragimov, (1992). *Russian Mathematical Surveys* 47 (4) 89-156.

V.A. Dorodnitsyn, R. Kozlov, S.V. Meleshko, P. Winternitz, (2018). *J. Phys. A: Math. Theor.* 51, 205202 (33pp).

M.O. Aibinu, S.C. Thakur, S. Moyo, (In Press). *Intl J. Nonl. Anal. & Appl.* https://arxiv.org/pdf/2106.03515.pdf

H.M. Baskonus, H. Bulut, Y. Pandir, (2014). *Int J Model Optim.* 4(4):305-309.

F.S. Costa, J.A.P.F. Marão, J.C.A. Soares, E C. de Oliveira, (2015). *J Math Phys.* 56: 033507.

Thank You!