Reduction type of hyperelliptic curves in terms of the valuations of their invariants.

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Elliptic curves reduction type

Let us consider the elliptic curve

$$E:\ y^2 = f(x) = (x-e_1)(x-e_2)(x-e_3) = x^3 + ax + b$$

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This model has **good reduction** at π if and only if their roots do not become equal modulo π since the **discriminant**

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Notice that $e_1 + e_2 + e_3 = 0$ and $a = e_1e_2 + e_2e_3 + e_3e_1$ and $b = e_1e_2e_3$.

Two such models define isomorphic elliptic curves if and only if $(a:b) = (a':b') \in \mathbb{P}^1_{2,3}$, or equivalently, if $(a^3:b^2) = (a'^3:b'^2) \in \mathbb{P}^1$, what happens if and only if $a^3/\Delta = a'^3/\Delta'$, i.e. if we have the j-**invariant** equality

$$j = j'$$

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$$y^2 = (x - e_1)(x - e_1 - a\pi^\alpha)(x - e_1 - b\pi^\beta) \text{ with } \alpha \leq \beta$$

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Make $X\pi^\alpha = (x - e_1)$ and $Y\pi^{3\alpha/2} = y$, then we get a new model:

$$Y^2 = X(X - a)(X - b\pi^{\beta-\alpha})$$

with at least two different roots modulo π.

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with at least two different roots modulo π.
So after maybe having to consider an extension we can always find a model with good reduction or bad reduction but with only two equal roots (a node).
More geometrically speaking: *we can always kill the cusps by doing blow-ups*.

Elliptic curves reduction type

Theorem (Tate)

Let $E/K : y^2 = x^3 + ax + b$ be an elliptic curve given by an integer model, i.e., $a, b \in \mathcal{O}_K$. Then:

- *E has* good *reduction if $v(\Delta(E)) = 0$.*
- *E has* multiplicative *(bad) reduction if $v(\Delta(E)) > 0$ and $v(a) = 0$.*
- *E has* additive *(bad) reduction if $v(\Delta(E)) > 0$ and $v(a) > 0$.*

After a finite field extension, additive reduction always becomes multiplicative or good. E has potentially good reduction *if and only if $3v(a) \geq v(\Delta(E))$, or equivalently, if $v(j(E)) \geq 0$.*

Elliptic curves: theta constants

We define the **theta constants** for a complex elliptic curve $E_\tau = \mathbb{C}/\Lambda$ with $\Lambda =< \tau, 1 >$ as:

$$\vartheta_{00} = \vartheta(0,\tau),\ \vartheta_{01} = \vartheta(1/2,\tau),\ \vartheta_{10} = \exp(\pi i\tau/4)\vartheta(\tau/2,\tau),\ \vartheta_{11} = 0.$$

Here $\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2\tau + 2\pi i n z)$.

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We have $a = -(\vartheta_{00}^8 + \vartheta_{01}^8 + \vartheta_{10}^8)$, $b = \frac{-1}{27}(\vartheta_{00}^4 + \vartheta_{01}^4)(\vartheta_{00}^4 + \vartheta_{10}^4)(\vartheta_{01}^4 - \vartheta_{10}^4)$ and $\Delta = \vartheta_{00}^8\vartheta_{01}^8\vartheta_{10}^8$. We say that they are **modular invariants**. Moreover:

$$\vartheta_{00}^4 = e_1 - e_2,\ \vartheta_{01}^4 = e_1 - e_3,\ \text{and } \vartheta_{02}^4 = e_3 - e_2.$$

★ Mumford gives a definition for theta constants based on line-bundles and the 2-torsion of an abelian variety that works for any field of characteristic different from 2.

★ Tate's results can be rewritten by saying that E has good reduction if and only if up to normalization the valuation of its 3 even theta constants is 0.

Stable reduction

- the reduction type depends on the model.
- in particular on the field of definition.
- we can blow-up singularities (maybe going to a bigger extension) to get models with nicer singularities.
- a model is called semi-stable if it is reduced and all singularities are at most double ones (nodes).
- there is a minimal semi-stable model (the stable model), where the components of genus 0 have to intersect other components at least in 3 points.

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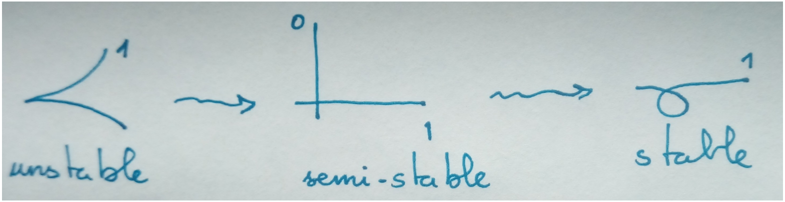


Genus 2 curves

Curves of genus 2 are **hyperelliptic**, meaning that they are given by a model:

$$y^2 = f(x) = (x - e_1)(x - e_2)(x - e_3)(x - e_4)(x - e_5)(x - e_6)$$

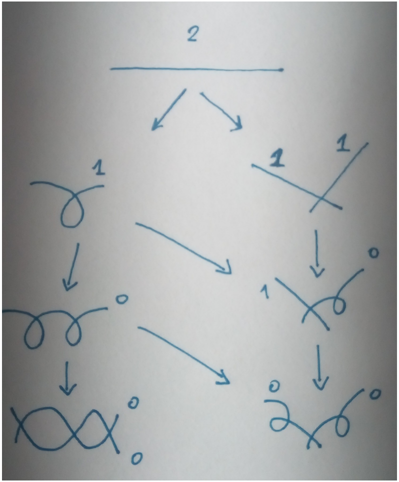
Their isomorphism classes are determined by the so-called **Igusa invariants**:

$$C \simeq C' \text{ if and only if } (J_2 : J_4 : J_6 : J_{10}) = (J_2' : J_4' : J_6' : J_{10}') \in \mathbb{P}^3_{1,2,3,5}$$

They are given by polynomial expressions on the coefficients of f but can also be written in terms of the differences of the roots of f (Clebs, Igusa, Mestre, etc). In addition they can be written in terms of the 10 even theta constants of genus 2 curves (hence they are **modular**!):

$$J_2 = \frac{\chi_{12}}{\chi_{10}}, J_4 = \alpha_4, J_6 = \alpha_6 \text{ and } J_{10} = \chi_{10}.$$

Reduction type of genus 2 curves



Reduction type of genus 2 curves

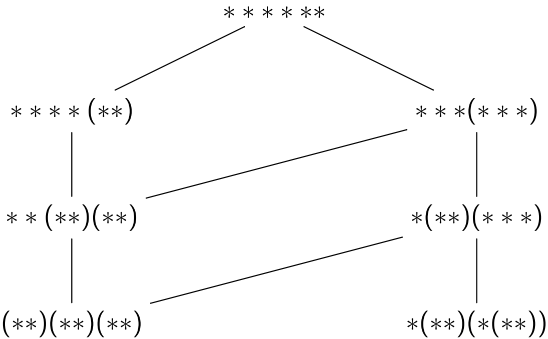
Theorem (Liu 92)

Let C/K be a smooth genus 2 curve, the reduction of the stable model of C is:

- *a smooth genus 2 curve if and only if $v(J_{2i}^5/J_{10}^i) \geq 0$ for all $i \leq 5$.*
- *irreducible with a single (doble) singular point if and only if $v(J_{2i}^6/J_{12}'^i) \geq 0$ for all $i \leq 5$ and $v(J_{10}^6/J_{12}'^5) > 0$.*
- *irreducible with two (double) singular point if and only if $v(J_{2i}^2/J_4'^i) \geq 0$ for all $i \leq 5$, $v(J_{10}^2/J_4'^5) > 0$, $v(J_{12}'/J_4'^3) > 0$ and $v(J_4/J_4') = 0$ or $v(J_6^2/J_4'^3) = 0$.*
- *2 projective lines intersecting each other at 3 points if and only if $v(J_{2i}^2/J_4'^i) > 0$ for all $2 \leq i \leq 5$.*
- *2 elliptic curves intersecting at 1 point if $v(J_4'^\epsilon/J_{2\epsilon}'^2) > 0$, $v(J_{10}^\epsilon/J_{2\epsilon}'^5) > 0$, $v(J_{12}'^2/J_{2\epsilon}'^6) > 0$, $v(J_4'^{3\epsilon}/(J_{10}^\epsilon J_{2\epsilon}')) = 0$ and $v(J_{12}'^\epsilon/(J_{10}^\epsilon J_{2\epsilon}')) = 0$.*
- *1 elliptic curve and a singular conic intersecting at 1 point if $v(J_4'^\epsilon/J_{2\epsilon}'^2) > 0$, $v(J_{10}^\epsilon/J_{2\epsilon}'^5) > 0$, $v(J_{12}'^2/J_{2\epsilon}'^6) > 0$, $v(J_4'^3/J_{12}') = 0$ and $v(J_{10}^\epsilon J_{2\epsilon}'/J_{12}'^\epsilon) > 0$.*
- *2 singular conics intersecting at one point otherwise.*

A different proof using cluster pictures

Those reduction types correspond to the following **cluster pictures**:



Semistable types of hyperelliptic curves by T. Dokchitser, V. Dokchitser, C. Maistret and A. Morgan.

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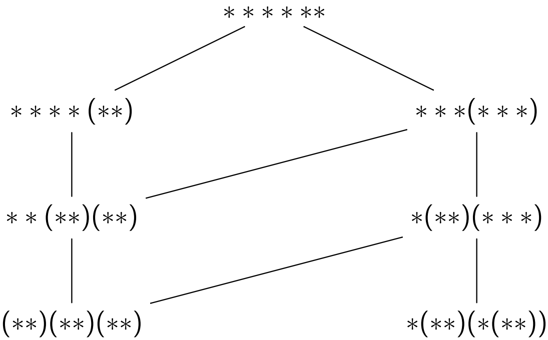

Semistable types of hyperelliptic curves by T. Dokchitser, V. Dokchitser, C. Maistret and A. Morgan.

★ In general determining the cluster picture is not easy!

A different proof

Let us first introduce the following invariants:

$$
\begin{aligned}
J'_{12} &= \sum (12, 3456)^3 (3456)^2 \\
J''_{6} &= \sum (12, 34)^2 (1234, 56)(56)^2 \\
A' &= \sum (12)^2 (34)^2 (56)^2 \\
B' &= \sum (123)^2 (456)^2
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Let us observe the following:

	D	J'_{12}	J''_6	B'	A'
******	$\neq 0$	-	-	-	-
****(**)	0	$\neq 0$	-	-	-
**(**)(**)	0	0	$\neq 0$	-	-
(**)(**)(**)	0	0	0	$\neq 0$	-
***(***)	0	0	0	0	$\neq 0$
***(*(**))	0	0	0	0	$\neq 0$
(**)((**))	0	0	0	0	$\neq 0$

A different proof

In order to distinguish the last three cases, we work in the compactification of $\mathcal{A}_2$ instead of in the one of $\mathcal{M}_2$, i.e. we forget about A' and we normalized again.

	J_{12}	${B'}^3 - {J'}_6^2$
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***(*(**))	0	$\neq 0$
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Remark: all this can also be seen in terms of theta constants.

Takase and Thomae formulas

Theorem (Thomae's formula)

For any subset S of B such that $\#S$ is even and $\#(U \circ S) = g + 1$, we have

$$\vartheta[\eta_S] = C \cdot (-1)^{\#U \circ S} \prod_{k \in U \circ S, l \notin U \circ S} (e_k - e_l)^{-1}.$$

here C is a constant independent of S.

Theorem (Takase)

For any disjoint decomposition $B = V \cap W \cap k, l, m$ with $\#V = \#W = g - 1$, we have

$$\frac{e_k - e_l}{e_k - e_m} = \epsilon(k, l, m) \left(\frac{\vartheta[U \circ (V \cap \{k, l\})]\vartheta[U \circ (W \cap \{k, l\})]}{\vartheta[U \circ (V \cap \{k, m\})]\vartheta[U \circ (W \cap \{k, m\})]} \right)^2.$$

Here $\epsilon(k, l, m) = \pm 1$.

Genus 3 curves

Genus 3 curves can be **hyperelliptic** and **non-hyperelliptic**. In this talk I will focus on hyperelliptic curves:

$$y^2 = f(x) = (x - e_1)...(x - e_8).$$

We have invariants describing the isomorphism classes in terms of the coefficients of f (**Shioda**), the differences of the roots (**Tsuyumine**) and in terms of theta constants (modular), and we have *passage formulas* from ones to the others [Lor].

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★ Can we generalized previous strategy for genus 3 in order to give a criterion to determine the reduction type of a hyperelliptic genus 3 curve in terms of their invariants?

Genus 3: clusters pictures

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- ******(**)
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- (**)(***)(***), (**)(***)(*(**)), (**)(*(**))(*(**))
- ****(****) and further degenerations.

Genus 3: exterior aspect

	Δ	I_{30}	I_{10}	I'_{10}	I_{12}	I_{20}	I'_{30}	I_8	I'_{20}	I_{18}	I_2
11111111	$\neq 0$	-	-	-	-	-	-	-	-	-	-
1111112	0	$\neq 0$	-	-	-	-	-	-	-	-	-
111122	0	0	$\neq 0$	-	-	-	-	-	-	-	-
11222	0	0	0	$\neq 0$	-	-	-	-	-	-	-
2222	0	0	0	0	$\neq 0$	-	-	-	-	-	-
111113	0	0	0	0	0	$\neq 0$	-	-	-	-	-
11123	0	0	0	0	0	0	$\neq 0$	-	-	-	-
1133	0	0	0	0	0	0	0	$\neq 0$	-	-	-
1223	0	0	0	0	0	0	0	0	$\neq 0$	-	-
233	0	0	0	0	0	0	0	0	0	$\neq 0$	-
11114	0	0	0	0	0	0	0	0	0	0	$\neq 0$
1124	0	0	0	0	0	0	0	0	0	0	$\neq 0$
224	0	0	0	0	0	0	0	0	0	0	$\neq 0$
134	0	0	0	0	0	0	0	0	0	0	$\neq 0$

Genus 3: a maximal cluster of size 3

The basic cluster picture with a size 3 cluster

$$* * * * * (* * *)$$

corresponds to a p.p.a.t. isomorphic to $A \times E$ with the product polarization.

Forgetting about the **curves invariants** and looking at **modular invariants** we get:

$$\begin{aligned}
\alpha_4 &= \psi_4 \otimes j_4 \\
\alpha_6 &= \psi_6 \otimes j_6 \\
\alpha_{10} &= \psi_{10} \otimes j_4 j_6 \\
\alpha_{12} &= \psi_{12} \otimes \Delta \\
\alpha'_{12} &= (81\psi_{12} + 16\psi_4^3 - \psi_6^2) \otimes \Delta + 12\psi_{12} \otimes (j_6^2 + 5j_4^3)
\end{aligned}$$

Re-using Tate's results and Liu's one we manage to distinguish the different cases.

Genus 3: a maximal cluster of size 4

The base case here is the cluster

$$* * * * (* * **)$$

that corresponds to two elliptic curves intersecting at 2 points. This is not a **compact reduction type**, so the previous strategy does not work.

We have

$$\begin{aligned} \alpha_4 &= j_4 \otimes j_4' \\ \alpha_6 &= j_6 \otimes j_6' \\ \gamma_{20}^2/\chi_{28} &= \Delta \otimes \Delta' \end{aligned}$$

But other products are not invariants of the genus 3 curve.

We may try brute force ... by playing with a model

$$y^2 = x(x - p\alpha_2)(x - p\alpha_3)(x - p\alpha_4)(x - 1)(x - \alpha_6)(x - \alpha_7).$$

★ Or trying a new strategy that may generalize to higher genus!

Theta constants to the rescue!

Proposition

The reduction type for genus 3 hyperelliptic curves is determined by the number of theta constants with given valuation.

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Proof.

The reduction type is determined by the cluster picture. It determines the valuations of the theta constants via Thomae's formula. We obtain different valuations configurations for each reduction type. □

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Lemma

The symmetric functions on the eighth powers of the thetas are Siegle modular forms.

Genus 3: the theorem

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The reduction type is determined by the valuation of the (eighth power of the) thetas who are computed with the Newton Polygon of the polynomial whose roots are all thetas. It only depends on the valuations of the coefficients of the polynomial. These coefficients are invariants. □

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Remark: those invariants may be huge ... but the criterion is very explicit!

Genus g curves

In order **to generalize** the previous ideas to any genus we will need a result stating that the reduction type is determined by the valuations of the thetas.

Already for non-hyperelliptic genus 3 curves I do not have a feeling about it being true. However, for hyperelliptic curves I do believe it:

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★ It should be possible to prove it using Takase and Thomae's formula in a clever way ...

Thanks!