PDEs with variable exponent

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Plan

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History of variable exponent spaces

- Variable exponent Lebesgue spaces appeared in the literature for the first time already in 1931 by W. Orlicz.

  W. Orlicz ; Uber konjugierte Exponentenfolgen. Studia Math., 3 : 200-211, 1931.

  In the paper of Orlicz, the following question is considered : Let (p_i) (with $p_i > 1$) and (x_i) be a sequences of real numbers such that $\sum_i x_i^{p_i}$ converges. What are the necessary and sufficient conditions on (y_i) for $\sum_i x_i y_i$ to converge ?

History of variable exponent spaces

It turns out that the answer is that $\sum_i (\lambda y_i)^{p_i'}$ should converge for some $\lambda > 0$ and $p_i' = \frac{p_i}{p_i - 1}$ which is essentially Holder's inequality in the space $l^{p(\cdot)}$.
Orlicz also considered the variable exponent function space $L^{p(\cdot)}$ on the real line, and proved the Holder inequality in the setting. However, after this one paper, Orlicz abandoned the study of variable exponent space, to concentrate on the theory of the function space that now bear his name.

J. Musielak and W. Orlicz. On modular spaces. Studia Math, 18 : 49-65, 1959.

Modular spaces

In the theory of Orlicz spaces, one defines the space L^φ to consist of those measurable functions

$$u : \Omega \to \mathbb{R}$$

for which

$$\rho(\lambda u) = \int_\Omega \varphi(\lambda |u(x)|) dx < \infty,$$

for some $\lambda > 0$.
φ has to satisfy certain conditions and $\Omega \subset \mathbb{R}^N$, $N \geq 1$.

Abstracting certain central properties of ρ, we are led to a more general class of so-called modular functions spaces which were first systematically studied by Nakano

H. Nakano. Modulared semi-ordered linear spaces. Maruzen Co. Ltd., Tokyo, 1950.

H. Nakano. Toplogy of linear topological spaces. Maruzen Co. Ltd., Tokyo, 1951.

In the appendix of the book of Nakano (1950), he mentioned explicitly variable exponent Lebesgue spaces as an example of the more general spaces he considers.

Modular functions

Following the work of Nakano, modular spaces were investigated by several people, most importantly by groups at Saporo (Japan), Voronezh (USSR) and Leifen (Netherlands). Somewhat later, a more explicit version of these spaces, modular function spaces, were investigated by Polish mathematicians, for instance, H. Hudzik, H. Kaminska and J. Musielak.

For a comprehensive presentation of modular function spaces, see the monograph by J. Musielak.

J. Musielak. Orlicz spaces and modular spaces, volume 1034 of Lecture Notes in Mathematics. Springer-Verlag, berlin, 1983.

Variable exponent Lebesgue spaces have been independently developed by Russian researchers, notably I. Sharapudinov. These investigations originated in a paper by I. Tsenov from 1961.

I. Tsenov. Generalization of the problem of best approximation of a function in the space l^s. Uch. Zap. Dagestan Gos. Univ., 7 :25-37, 1961.

and, were briefly touched on by V. Portnov.

V. R. Portnov. Certain properties of the Orlicz spaces generated by the functions $M(x, w)$. Dokl. Akad. nauk SSSR, 170 : 1269-1272, 1966

V. R. Portnov. On the theory of Orlicz spaces which are genrated by variable N-functions. Dokl. Akad. Nauk SSSR, 175 : 296-299, 1967.

Luxemburg norm

The question raised by I. Tsenov and solved by I. Sharapudinov is the minimization of

$$\int_a^b |u(x) - v(x)|^{p(x)} dx,$$

where u is a fixed function and v varies over a finite dimensional subspace of $L^{p(\cdot)}([a,b])$.
I. Sharapudinov also introduced the Luxemburg norm for the Lebesgue Space and showed that this space is reflexive if the exponent satisfies $1 < p^- \leq p^+ < \infty$.

I.I. Sharapudinov. On the topology of the space $L^{p(t)}([0,1])$. Math. Notes, 26 (3-4) : 796-806, 1976.

I.I. Sharapudinov. Approximation of functions in the metric of the space $L^{p(t)}([a,b])$ and quadrature formulas. (Russian). In constructive function theory'81 (Varna, 1981), page 189-193. Publ. House Bulgar. Acad. Sci., Sofia, 1983.

I.I. Sharapudinov. The basis property of the Haar system in the space $L^{p(t)}([0,1])$ and the principle of localization in the mean, (Russian). Math. Sb. (N.S.), 130 (172) : 275-283, 286, 1986.

In the mid-80', V. Zhikov started a new line of investigation, that was th become intimately related to the study of variable exponent spaces, considering variational integrals with non-standard growth conditions. Another early PDE paper is done by O. Kovacik, but this paper appears to have had little influence on later developments.

V.V. Zhikov. Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR-Izv, 29 : 675-710, 877, 1987.

O. Kovacik. Parabolic equations in generalized Sobolev spaces $W^{k,p(x)}$. Fasc. Math., 25 : 87-94, 1995.

Major step for variable exponent spaces

The next major step in the investigation of variable exponent spaces was the paper by O. Kovacik and J. Rakosnik in the 90's.

O. Kovacik and J. Rakosnik. On the spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czechoslovak Math J., 41(116) : 592-618, 1991.

This paper established many of the basic properties of Lebesgue and Sobolev spaces with variable exponent in $\mathbb{R}^n$.
During the following ten years, there were many scattered efforts to understand these spaces.

At the turn of the millennium, various developments lead to the start of a period of systematic intense study of variable exponent spaces : First, the connection was made between variable exponent spaces and variational integrals with non-standard growth and coercivity condition. It was also observed that these non-standard variational problems are related to modeling of so called electrorheological fluids. Later on, other applications have emerged in thermorheological fluids and image processing.

E. Acerbi and G. Mingione ; Regularity results for a class of functionals with non-standard growth. Arch. Ration. Mech. Anal., 156 : 121-140, 2001.

V.V. Zhikov ; On Lavrentiev's phenomenon. Rus. J. Math. Phys, 3 : 249-269, 1995.

K.R. Rajagopal and M. Ruzicka ; Mathematical modeling of electrorheological materials. Const. Mech. and Thermodynamics, 13 : 59-78, 2001.

- K.R. Rajagopal and M. Ruzicka ; On the modeling of electrorheological materials.
- M. Ruzicka ; Modeling and mathematical theory, volume 1748 of lecture notes in mathematics. Springer-verlag, Berlin, 2000.
- Variable exponent functionals in image restoration. Applied Mathematics and Computation, 216(3) : 870-882, 2010.
- R. Aboulaich, D. Meskine and A ; Souissi. New diffusion models in image processing. Comput. Math. Appl, 56 : 874-882, 2008.
- S. Antontsev and J. F. Rodrigues. On stationary thermo-rheological viscous flows. Ann. Univ. Ferrara. Sez. VII. Sci. Mat., 52(1) : 19-36, 2006.
- E. M. Bollt, R. Chartrand, S. Esedoglu, P. Schultz and K.R. Vixie. Graduated adaptative image denoising : local compromise between total variation and isotropic diffusion. Adv. Comput. Math, 31 (1-3) : 61-85, 2009.
- Y. Chen, S. Levine and M. Rao. Variable exponent linear growth functionals in image restaoration. SIAM J. Appl. Math., 66 : 1383-1406, 2006.

Even more important thing is the fact that the “correct” condition for regularity of variable exponents was found. This condition, which we call log-Hölder continuity, was used by L. Diening to show that the maximal operator is bounded on $L^{p(\cdot)(\Omega}$ when Ω is bounded. He also showed that the boundedness holds in $L^{p(\cdot)}(\mathbb{R}^N)$ if the exponent is constant outside a compact set. The case of unbounded domains was soon improved by D. Cruz-Uribe, A. Fiorenta and C. Neugebauer and, independently, A ; Nekvinda, so that a decay condition replaces the constancy at infinity. The boundedness of the maximal operator open us the door for treating a plethora of other operators. For instance one can then consider the Riesz potential operator and thus prove Sobolev embeddings. Such results indeed followed in quick succession starting from the middle of 2000. The boundedness of the maximal operator and other operators is a subtle question and improvements of these initial results have been made since then in many papers.

L. Diening. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. Math. Inequal. Appl, 7 : 245-253, 2004.

D. Cruze-Uribe, A. Fiorenza and C. Neugebauer. The maximal function on variable L^P spaces. Ann. Acad. Sci. Fenn. Math., 28 : 223-238 ; 29(2004), 247-249, 2003.

A. Nekvinda. Maximal operator on variable Lebesgue spaces for almost monotone radial exponent. J. Math. Anal. Appl., 337 :1345-1365, 2008.

Plan

The Lebesgue space $\Phi - \mathit{function}$

Definition 1

Let $(A, \sum, \mu)$ be a σ-finite, complete measure space. We define $\mathcal{P}(A, \mu)$ to be the set of all measurable functions $p : A \to [1, \infty]$. Functions $p \in \mathcal{P}(A, \mu)$ are called variable exponents on A.
We define $p^- := essinf_{y \in A} p(y)$ and $p^+ := esssup_{y \in A} p(y)$.
If $p^+ < \infty$, then we call p a bounded variable exponent.
If $p \in \mathcal{P}(A, \mu)$, then we define $p' \in \mathcal{P}(A, \mu)$ by $\frac{1}{p(y)} + \frac{1}{p'(y)} = 1$, for all $y \in A$, where $\frac{1}{\infty} := 0$.
The function p' is called the dual (or conjugate) variable exponent of p. In the special case where μ is the n-dimensional Lebesgue measure and Ω is an open subset of $\mathbb{R}^N$, we denote $\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu)$.

Basics properties

Definition 2

The Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_p(.)(u) := \int_\Omega |u|^{p(x)} dx$$

is finite.

Definition 3

If the exponent is bounded (if $p^+ < \infty$), then the expression $|u|_{p(.)} := \inf \left\{ \lambda > 0 / \rho_{p(.)}(\frac{u}{\lambda}) \leq 1 \right\}$ define a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm.

A function $x_M := \inf\left\{\lambda > 0/ \int_\Omega M(\frac{x(t)}{\lambda})dx \leq 1\right\}$, where $M(u)$ is an even function that increases for positive u, $\lim_{u\to 0}\dfrac{M(u)}{u} = \lim_{u\to 0}\dfrac{u}{M(u)} = 0$, $M(u) > 0$ for $u > 0$ and G is a bounded set in $\mathbb{R}^N$, is called the Luxemburg norm because this norm where studied by W. A. J. Luxemburg in 1955.

W. A. J. Luxemburg. Banach function spaces. T. U. Delft (1955), Thesis.

Proposition 3.1

The space $(L^{p(.)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < \infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Hölder type inequality

Proposition 3.2

$$|\int_\Omega uvdx| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(.)}|v|_{p'(.)},$$

for all $u \in L^{p(.)}(\Omega)$ *and* $v \in L^{p'(.)}(\Omega)$.

Relationship between Luxemburg norm and modular function

An important role in manipulating the Lebesgue spaces with variable exponents is played by the modular $\rho_{p(.)}$ of the space $L^{p(\cdot)}(\Omega)$.

Lemma 4

If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then the following properties hold :

$$(i)|u|_{p(.)} > 1 \Rightarrow |u|_{p(.)}^{p^-} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^+};$$

$$(ii)|u|_{p(.)} < 1 \Rightarrow |u|_{p(.)}^{p^+} \leq \rho_{p(.)}(u) \leq |u|_{p(.)}^{p^-};$$

$$(iii)|u|_{p(.)} < 1(resp = 1; > 1) \Leftrightarrow \rho_{p(.)}(u) < 1(resp = 1; > 1);$$

$$(iv)|u_n|_{p(.)} \to 0(resp \to \infty) \Leftrightarrow \rho_{p(.)}(u_n) \to 0(resp \to \infty);$$

Embeddings results

It is well known from the theory of Classical Lebesgue spaces (Lebesgue spaces with constant exponent) that $L^p(\Omega)$ is a subspace of $L^q(\Omega)$ with $p, q \in [1, \infty]$ if and only if $p \geq q$ and $\mu(\Omega) < \infty$. This suggests that a similar condition characterizes the embedding $L^{p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$, for $p, q \in \mathcal{P}(\Omega)$. Naturally, this question is closely related with the H́older type inequality. Recall that the norm of the embedding $L^{p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is the smallest constant $K > 0$ for which $\|f\|_{p(.)} \leq K\|f\|_{q(.)}$.

Embeddings results

Theorem 5

Let $p, q \in \mathcal{P}(\Omega)$*. Define the exponent* $r \in \mathcal{P}(\Omega)$ *by* $\frac{1}{r(y)} := \max\left\{\frac{1}{q(y)} - \frac{1}{p(y)}, 0\right\}$ *for all* $y \in \Omega$*. If* $q \leq p$*,* μ*-almost everywhere and* $1 \in L^{r(\cdot)}(\Omega)$*, then* $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *with norm at most* $2\|1\|_{L^{r(\cdot)}(\Omega)}$.

Theorem 6

Let $p, q, r \in \mathcal{P}(\mathbb{Z}^n)$*, with* $\frac{1}{r(x)} := \max\left\{\frac{1}{q(x)} - \frac{1}{p(x)}, 0\right\}$ *and* $1 \in L^{r(\cdot)}(\mathbb{Z}^n)$*. Then* $l^{p(\cdot)}(\mathbb{Z}^n) \hookrightarrow L^{q(\cdot)}(\mathbb{Z}^n)$.

Theorem 7

If $p \in \mathcal{P}(\Omega)$ *with* $p < \infty$*, then* $C_0^\infty(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

Plan

The Sobolev space is a vector space of functions with weak derivatives. One motivation os studying these spaces is that solutions of PDEs belong naturally to Sobolev spaces.
Let $\Omega \subset \mathbb{R}^N$ be an open set. We start by recalling the definition of weak derivatives.

Definition 8

Assume that $u \in L^1_{loc}(\Omega)$. Let $\alpha := (\alpha 1, ..., \alpha_N) \in \mathbb{N}^N$ be a multi-index. If there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u \frac{\partial^{\alpha_1+\ldots+\alpha_N}\Psi}{\partial^{\alpha_1}x_1\ldots\partial^{\alpha_N}x_N}dx = (-1)^{\alpha_1+\ldots+\alpha_N}\int_\Omega \Psi g dx,$$

for all $\Psi \in C_0^\infty(\Omega)$, then g is called a weak partial derivative of u with respect to α. The function g is the denoted by $\partial_\alpha u$ or by $\frac{\partial^{\alpha_1+\ldots+\alpha_N}u}{\partial^{\alpha_1}x_1\ldots\partial^{\alpha_N}}$. Moreover, we write u to denote the weak gradient $(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N})$ of u and we write short $\partial_j u$ for $\frac{\partial u}{\partial x_j}$ with $j = 1, ..., N$.. More generally, we write ${}^k u$ to denote the tensor with entries $\partial_\alpha u$, $|u| = k$.

Definition 9

The function $u \in L^{p(.)}(\Omega)$ belongs to the space $W^{k,p(.)}(\Omega)$, where $k \in \mathbb{N}$ and $p \in \mathcal{P}(\Omega)$, if its weak partial derivatives $\partial_\alpha u$ with $|\alpha| \leq k$ exist and belongs to $L^{p(.)}(\Omega)$.

Definition 10

We define the semimodular on $W^{k,p(.)}(\Omega)$ by

$$\rho_{k,p(.)}(u) := \sum_{0 \leq |\alpha| \leq k} \rho_{p(.)}(\partial_\alpha u),$$

which induces a norm by

$$\|u\|_{k,p(.)} := \inf \left\{ \lambda > 0 / \rho_{k,p(.)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

For $k \in \mathbb{N}$, the space $W^{k,p(.)}(\Omega)$ is called Sobolev space with variable exponent and its elements are called Sobolev functions.

Definition 11

A function u belongs to $W^{k,p(.)}_{loc}(\Omega)$ if $u \in W^{k,p(.)}(U)$ for every compact set $U \subset \Omega$. We equip $W^{k,p(.)}_{loc}(\Omega)$ with the initial topology induced by the embeddings $W^{k,p(.)}_{loc}(\Omega) \hookrightarrow W^{k,p(.)}(U)$, for all compact set $U \in \Omega$

Theorem 12

Let $p \in \mathcal{P}(\Omega)$. The space $W^{k,p(.)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive and uniformly convex if $1 < p^- \leq p^+ < \infty$.

Lemma 13

Let $p \in \mathcal{P}(\Omega)$. Then, $W^{k,p(.)}(\Omega) \hookrightarrow W^{k,p^-}_{loc}(\Omega)$. If $|\Omega| < \infty$, then $W^{k,p(.)}(\Omega) \hookrightarrow W^{k,p^-}(\Omega)$.

We now defines Sobolev spaces with zero boundary values and given basics properties for them.

Definition 14

Let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. The Sobolev space $W_0^{k,p(.)}(\Omega)$ with zéro boundary values is the closure of the set of $W^{k,p(.)}(\Omega)$-functions with compact support.

Theorem 15

Let $p \in \mathcal{P}(\Omega)$. The space $W_0^{k,p(.)}(\Omega)$ is a Banach space, which is separable if p is bounded, and reflexive and uniformly convex if $1 < p^- \leq p^+ < \infty$.

Particular case where $k = 1$

For a measurable function $u :\to \mathbb{R}$, we introduce the following notation :

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)} dx + \int_\Omega |\nabla u|^{p(x)} dx.$$

Then we have the following lemma.

Lemma 16

If $u \in W^{k,p(.)}(\Omega)$ the the following properties hold :

$$(i) |u|_{1,p(.)} > 1 \Rightarrow |u|^{p^-}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq |u|^{p^+}_{1,p(.)};$$

$$(ii) |u|_{1,p(.)} < 1 \Rightarrow |u|^{p^+}_{1,p(.)} \leq \rho_{1,p(.)}(u) \leq |u|^{p^-}_{1,p(.)};$$

$$(iii) |u|_{1,p(.)} < 1 (resp = 1; > 1) \Leftrightarrow \rho_{1,p(.)}(u) < 1 (resp = 1; > 1);$$

$$(iv) |u_n|_{1,p(.)} \to 0 (resp \to \infty) \Leftrightarrow \rho_{1,p(.)}(u_n) \to 0 (resp \to \infty);$$

Sobolev-Poincaré inequalities and embeddings

In this part, we assume that the exponent p is $\log$-Hölder continuous with $1 \leq p^- \leq p^+ < n$.

Definition 17

We say that a function $\alpha : \Omega \to \mathbb{R}$ is locally $\log$-Hölder continuous on Ω if there exists $C_{1>0}$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{C_1}{log(e + \frac{1}{|x-y|})},$$

for all $x, y \in \Omega$.
We say that α satisfies the $\log$-Hölder decay condition if there exist α_∞ and a constant C_2 such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_2}{log(e + |x|)},$$

for all $x \in \Omega$.

We say that α is globally $\log$-Hölder continuous in Ω if it is locally $\log$-Hólder continuous and satisfies the $\log$-Hölder decay condition.
The constants C_1 and C_2 are called the local $\log$-Hölder constant and the $\log$-Hölder decay constant, respectively. The maximum $\max\{C_1, C_2\}$ is just called the $\log$-Hölder constant of α.

Definition 18

We define the following class of variable exponents

$$\mathcal{P}^{log}(\Omega) := \left\{ p \in \mathcal{P}(\Omega) / \frac{1}{p} \text{ is globally } \log\text{-Hölder continuous} \right\}.$$

By $c_{log}(p)$ or c_{log}, we denote the $\log$-Hölder constant of $\frac{1}{p}$.

Definition 19

A bounded domain $\Omega \subset \mathbb{R}^N$ is called an α-John domain, $\alpha > 0$, if there exists $x_0 \in \Omega$ (the John center) such that each point in Ω can be joined to x_0 by a rectifiable path γ (The John path) parametrized by arc-length such that $B(\gamma(t), \frac{1}{\alpha}t) \subset \Omega$, for all $t \in [0, l(\gamma)]$, where $l(\gamma)$ is the length of γ. The ball $B(x_0, \frac{1}{2\alpha} diam(\Omega))$ is called the John ball.

Definition 20

We define the Sobolev conjugate exponent point-wise, i.e.,

$$p^\star(x) := \frac{np(x)}{n - p(x)},$$

when $p(x) < n$
and $p^\star(x) = \infty$, otherwise.

Let $p \in \mathcal{P}^{log}(\Omega)$ satisfy $1 \leq p^- \leq p^+ < n$.
(a) For every $u \in W_0^{1,p(.)}(\Omega)$,

$$\|u\|_{L^{p^\star(.)}(\Omega)} \leq c\|\nabla u\|_{L^{p(.)}(\Omega)},$$

for $u \in W^{1,p(.)}(\Omega)$. The constant c depends only on the dimension n, α, $c_{log}(p)$ and p^+.
(b) If Ω is a bounded α-John domain, then

$$\|u - \langle u\rangle_\Omega\|_{L^{p^\star(.)}(\Omega)} \leq \|\nabla u\|_{L^{p(.)}(\Omega)},$$

for $u \in W^{1,p(.)}(\Omega)$. The constant c depends only on the dimension n, α, $c_{log}(p)$ and p^+.

Corollary 21

Let Ω be a bounded α-John domain and let $p \in \mathcal{P}^{log}(\Omega)$. Let $q \in \mathcal{P}(\Omega)$ be bounded and assume that $q \leq p^\star$. Then

$$W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega),$$

where the embedding constant depends only on α, $|\Omega|$, n, $c_{log}(p)$ and q^+.

Theorem 22

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $p \in \mathcal{P}^{log}(\Omega)$. Then,

$$W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega) \text{ (compact embedding)}.$$

Corollary 23

Let Ω be a bounded domain and let $p \in \mathcal{P}^{log}(\Omega)$ satisfy $p^+ < n$. Then

$$W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)^\star - \epsilon}(\Omega)$$

, for every $\epsilon \in (0, n')$, where n' is such that $\frac{1}{n} + \frac{1}{n'} = 1$.

Plan

Motivation

1- Mathematical motivation

The first motivation of using Lebesgue and Sobolev spaces with variable exponent to solve PDEs was the following (done by Kovacik and Rakosnik) :
Consider the nonlinear Dirichlet boundary value problem.

$$\sum_{|\alpha|\leq k}(-1)^{|\alpha|}D^{\alpha}a_{\alpha}(x,\delta_k u)=f \text{ on } \Omega, \tag{5.1}$$

$$u=0 \text{ on } \partial\Omega, \tag{5.2}$$

where $\delta_k u=\{D^{\alpha}u : |\alpha|\leq k\}$.
One of the common approaches to the weak solvability of the problem (5.1)-(5.2) is based on the Browder theorem and assumes that the following Leray-Lions Conditions are satisfied.

O. Kovacik and J. Rakosnik. On the spaces $L^{p(x)}$ and $W^{1,p(x)}$. Czechoslovak Math J., 41(116) : 592-618, 1991.

J. Leray and J. L. Lions. Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. Bulletin de la S. M. F., tome 93 (1965), p. 97-107.

$$|a_\alpha(x,\xi)| \leq g(x) + c \sum_{|\alpha|\leq k} |\xi_\alpha|^{p-1} \text{ growth conditions,} \tag{5.3}$$

with $g \in L^{p'}(\Omega)$ and

$$\sum a_\alpha(x,\xi)\xi_\alpha \geq c_1 \sum_{|\alpha|\leq k} |\xi_\alpha|^p - c_2 \text{ coercivity conditions} \tag{5.4}$$

with some $p \in (1,\infty)$.

It is then natural to look for a weak solution in the Sobolev space $W_0^{1,p(.)}(\Omega)$. Consider a more general situation, when $\Omega = \Omega_1 \cup \Omega_2$, $1 < p_1 < p_2 < \infty$, and the conditions (5.3)-(5.4) are satisfied with p_i on Ω_i. If we simply use the above scheme to find the weak solution of (5.1)-(5.2) in $W_0^{k,p}(\Omega)$, we see that the validity of conditions (5.3) and (5.4) requires $p = \min\{p_1, p_2\}$ and $p = \max\{p_1, p_2\}$ respectively. Therefore, the common way is that p has to vary, a function of $x \in \Omega$.

2- Physical motivation

The interest of the study of Lebesgue and Sobolev spaces with variable exponent lies on the fact that most materials can be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$, where p is a fixed constant, but for some materials with inhomogeneities, for instance electrorheological fluids (sometimes referred to as “smart fluids"), this is not adequate, but rather the exponent p should be able to vary. These fluids are smart materials which are concentrated suspensions of polarizable particles in a non-conducting dielectric liquid. By applying an electric field, the viscosity can be changed by a factor up to 10^5, and the fluid can be transformed from liquid state into semi-solid stats within milliseconds. The process is reversible. As example of electrorheological fluids, we have alumina Al_2O_3 particles.
Note also that by replacing p by $p(x)$ on the models used to debluring and denoising images, one gets a powerful and faster denoising process.

We consider the following nonlinear boundary value problem :

$$-div(a(x, \nabla u)) = f(x, u) \text{ in } \Omega \tag{5.5}$$

$$u = 0 \text{ on } \partial\Omega, \tag{5.6}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain with smooth boundary.
The existence and uniqueness of weak and entropy solutions of problem (5.5)-(5.6) was done by Ouaro and Traoré.

S. Ouaro and S. Traoré. Weak and entropy solutions to nonlinear elliptic problems with variable exponent. J. Convex Anal. 16 (2009), N°2, 523-541.

Assumptions on the data

(H_1) :f$\in L^{\infty}(\Omega)$;
(H_2) :f$\in L^{1}(\Omega)$;
(H_3) :f is a Carathéodory function that is f$\Omega \times \mathbb{R} \to \mathbb{R}$ sucht that $f(.,t)$ is measurable and $f(x,.)$ is continuous.
(H_4) :p(.) : $\Omega \to \mathbb{R}$ is a measurable function such that $1 < p- \leq p^{+} < \infty$.
(H_5) :a(x,ξ) : $\Omega \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{N}$ is the continuous derivative with respect to ξ of the mapping $A : \Omega \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ i.e. $a(x,\xi) = \nabla_{\xi} A(x,\xi)$ such that :
a) $A(x,0) = 0$, for $a.e.x \in \Omega$
b) There exists $C_{1>0}$ such that

$$|a(x,\xi)| \leq C_1(j(x) + |\xi|^{p(x)-1}),$$

for $a.e.x \in \Omega$ and for every $\xi \in \mathbb{R}^{N}$, where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$.
c) $(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0$, for $a.e.x \in \Omega$ and for every $\xi \in \mathbb{R}^{N}$ with $\xi \neq \eta$.
d) $|\xi|^{p(x)} \leq a(x,\xi).\xi \leq p(x)A(x,\xi)$, for $a.e.x \in \Omega$ and for every $\xi \in \mathbb{R}^{N}$.

As examples of models with respect to above assumptions, we can give the following :
(i) Set $A(x,\xi) = (\frac{1}{p(x)})|\xi|^{p(x)}$, $a(x,\xi) = |\xi|^{p(x)-2}\xi$, where $p(x) \geq 2$. Then we get the $p(x)$-Laplace operator $div(|\nabla u|^{p(x)-2}\nabla u)$.
(ii) Set $A(x,\xi) = (\frac{1}{p(x)})((1+|\xi|^2)^{\frac{p(x)}{2}} - 1)$, $a(x,\xi) = (1+|\xi|^2)^{\frac{p(x)-2}{2}}\xi$, where $p(x) \geq 2$. Then we obtain the generalized mean curvature operator $div((1+|\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u)$.

Weak energy solution for $f \in L^\infty(\Omega)$

Definition 24

A weak solution of problem (5.5)-(5.6) is a function $u \in W_0^{1,1}(\Omega)$ such that $a(.,\nabla u) \in (L^1_{loc}(\Omega))^N$ and $\int_\Omega a(x,\nabla u).\nabla\varphi dx = \int_\Omega f(x)\varphi dx$, for all $\varphi \in C_0^\infty(\Omega)$.
A weak energy solution is a weak solution such that $u \in W_0^{1,p(.)}(\Omega)$.

Theorem 25

Assume (H_1), (H_4), (H_5). *Then ; there exists a unique weak energy solution of* $(5.5)-(5.6)$.

Proof of Theorem 25 We define the energy functional

$$I(u) = \int_\Omega A(x, \nabla u)dx - \int_\Omega fudx.$$

We prove that $I \in C^1(W_0^{1,p(.)}(\Omega), \mathbb{R})$, bounded from below, coercive and weakly lower semi-continuous with the derivative given by

$$\langle I'(u), \varphi \rangle = \int_\Omega a(x, \nabla u).\nabla \varphi dx - \int_\Omega f \varphi dx,$$

for all $u, \varphi \in W_0^{1,p(.)}(\Omega)$.
We also use assumption $(H)_5$ and the Poincaré inequality since $W_0^{1,p(.)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega)$, to prove the uniqueness of weak energy solution.

Weak solutions for $f(x,u)$

In this part, we study problem (5.5)-(5.6) for a Carathéodory function f.
Let

$$F(x,t) = \int_0^t f(x,s)ds$$

We assume that :
(H_6) : There exists $C_1 > 0$ such that $|f(x,t)| \leq c_1 + c_2|t|^{\beta-1}$, where $1 \leq \beta < p_-$.
We have the following result.

Theorem 26

Under assumptions (H_3), (H_4), (H_5) and (H_6), the problem (5.5)-(5.6) has at least one weak energy solution.

Proof of theorem 26

We define the functional

$$I(u) = \int_\Omega A(x, \nabla u)dx - \int_\Omega F(x, u)dx.$$

We prove that I is bounded from below, coercive, lower semi-continuous and in $C^1(W_0^{1,p(.)}(\Omega), \mathbb{R})$, to get the result of at least one weak energy solution of problem (5.5)-(5.6).

Assume now that $F^+(x,t) = \int_0^t f^+(x,s)ds$, is such that there exists $C_1 > 0$, $C2 > 0$ such that
(H_7) : $|f^+(x,t)| \leq C_1 + C_2|t|^{\beta-1}$, where $1 \leq \beta < p^-$.
We have the following result.

Theorem 27

Under assumptions (H_3), (H_4), (H_5) and (H_7), the problem (5.5)-(5.6) has at least one weak energy solution.

Proof. As $f = f^+ - f^-$, then $I(u) \geq \int_\Omega A(x,\nabla u)dx - \int_\Omega F^+(x,u)dx$. Therefore, as in the proof of theorem above, the result follows.

There is no uniqueness of weak solution of problem (5.5)-(5.6) when the right hand side term is Carathéodory. Indeed the function

$$f(x,t) = \lambda(t^{\gamma-1} - t^{\beta-1}), \tag{5.7}$$

where $1 < \beta < \alpha < p^-$ and $\lambda > 0$ verify (H_6) and (H_7). Mihailescu and Radulescu proved that with (5.7), the problem (5.5)-(5.6) has at least two distinct non negative non trivial weak energy solutions.

M. Mihailescu, V. Radulescu. A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. R. Soc. 462 (2006), 2625-2641.

Entropy solution for $f \in L^1(\Omega)$

As the right-hand side of problem (5.5)-(5.6) belongs in $L^1(\Omega)$, the suitable notion of solution for the study of the problem is the notion of entropy solution. See following reference.

Ph. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vazquez. An L^1-Theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Sc. Norm. Super. Pisa, Cl. Sci, IV. Ser. 22 (1995), 241-273.

Definition 28

A measurable function u is an entropy solution of problem (5.5)-(5.6) if, for every $t > 0$, $T_t(u) \in W_0^{1,p(.)}(\Omega)$ and

$$\int_\Omega a(x, \nabla u).\nabla T_t(u - \varphi)dx \leq \int_\Omega f(x)T_t(u - \varphi)dx,$$

for all $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

The troncation function T_t is defined by $T_t(s) := \max\{-t, \min(t, s)\}$.

Theorem 29

Assume (H_2), (H_4), (H_5). Then there exists a unique entropy solution u to problem (5.5)-(5.6).

Proof 1- Uniqueness. By monotonicity assumptions, Sobolev embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow W_0^{p^-}(\Omega)$ and Poincar é inequality in constant exponent, we get that if u and v are entropy solutions of problem (5.5)-(5.6),

$$\int_\Omega |T_t(u-v)|^{p^-}\,dx \le \int_\Omega |\nabla(T_t(u-v))|^{p^-}\,dx = 0,$$

for all $t > 0$. Hence $u = v$ a.e. in Ω.
2- Existence. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of bounded functions, strongly converging to $f \in L^1(\Omega)$ and such that $\|f_n\|_1 \le \|f\|_1$, for all $n \in \mathbb{N}$. As $f_n \in L^\infty(\Omega)$ then problem (5.5)-(5.6) with n as a unique sequence of weak energy solution $(u_n)_{n\in\mathbb{N}}$.

(1) $u_n \to u$ in measure and a.e. in Ω.
(2) ∇u_n converges in measure to the weak gradient of u.
(3) $a(x, \nabla u_n)$ converges to $a(x, \nabla u)$ strongly in $L^1(\Omega)^N$.
After passing to the limit as $n \to \infty$ in

$$\int_\Omega a(x, \nabla u_n).\nabla(T_t(u_n - \varphi))dx = \int_\Omega fT_t(u_n - \varphi)dx,$$

by using Lebesgue's theorem and Fatou's lemma, it follows that

$$\int_\Omega a(x, \nabla u).\nabla(T_t(u - \varphi))dx \leq \int_\Omega fT_t(u - \varphi)dx$$

Thank You for Your Kind Attention