Complex contact manifolds and holomorphic Jacobi manifolds.

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Introduction

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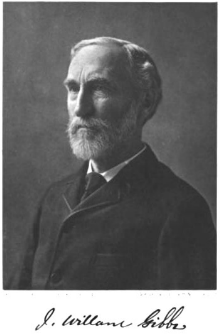
- The concept of a contact structure first appeared in 1896 in Sophus Lie's work. See his monograph "Geometrie der Berührungstransformationen" (The geometry of contact transformation).

Introduction

- In Gibbs' work (1873) contact structures appeared as a geometric framework for formulating thermodynamics laws.



Introduction

- Contact manifolds naturally arise in Hamiltonian mechanics. Indeed, a natural contact structure arises on any level submanifold of the Hamiltonian function, defined in the phase space of a mechanical system.

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- The modern perception of contact manifolds should be accredited to G. Reeb. In his work entitled "Sur certaines propriétés topologiques des trajectoires des systèmes dynamiques", Reeb referred to a contact manifold as "système dynamique avec invariant intégral de Monsieur Elie Cartan".

V. Arnold said: All geometry is contact geometry

Contact geometry is closely related to other areas of mathematics and physics such as:

- Symplectic geometry;
- Fluid mechanics, Quantum Physics, String theory;
- Knot theory, Riemannian geometry.
- V. Arnold said "All geometry is contact geometry"

Basic definitions

Definition: A **real contact structure** on a $(2n+1)$-dimensional smooth manifold M is a maximally non-integrable hyperplane field ξ.

• Any maximally non-integrable hyperplane field on M is locally defined as the kernel of a 1-form α satisfying: $\alpha \wedge (d\alpha)^n \neq 0$.

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Remark: Observe that if α is globally defined on M then $\alpha \wedge (d\alpha)^n$ is a volume form. In this case, M is orientable.

Example

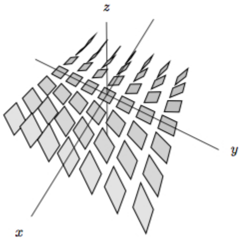

Figure: Contact structure on $\mathbb{R}^3$ given by the kernel of $\alpha = dz + xdy$.

- ⋆ At $(x, 0, 0)$, 2-plane field is spanned by $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\}$.
- ⋆ All contact 3-dimensional manifolds looks locally like this one.

More Examples

Example

• On $\mathbb{R}^{2n+1}$ with the standard coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$, the 1-form

$$\alpha = dz + \sum_{i=1}^{n} x_i dy_i$$

is a contact 1-form corresponding to $\xi = \ker(\alpha)$.

• Consider the unit sphere $\mathbb{S}^{2n+1}$ of $\mathbb{R}^{2n+2}$ with the cartesian coordinates $(x_1, \cdots, x_{n+1}, y_1, \cdots, y_{n+1})$. A contact 1-form on the unit sphere $\mathbb{S}^{2n+1}$ is: $\alpha_0 = \sum_{i=1}^{n+1}(x_i dy_i - y_i dx_i)$.

Contact and Poisson geometry

- Contact geometry is the framework for classical thermodynamics while Poisson geometry is a natural framework for classical mechanics. These two geometries are closely related.

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- It's known that Hamilton's equations are underpinning of classical mechanics. But it turns out that Hamilton's equations are almost "the same as" Maxwell relations, up to change of the names of the variables.
- The resemblance between Hamilton's equations and Maxwell's relations is not surprising since contact geometry is closely related to symplectic geometry and we know that symplectic structures are special cases of Poisson structures. Furthermore, the method of Legendre transformations plays an important role in both classical mechanics and thermodynamics.

Poisson geometry

Definition:

- A Poisson algebra is a commutative associative algebra $\mathcal{A}$ over $\mathbb{R}$ together with a Lie algebra bracket $\{\ ,\ \}$ for which each operator $X_h = \{\ , h\}$ is a derivation of the associative algebra structure.
- In particular, if $\mathcal{A}$ is the algebra $C^\infty(M)$ of smooth functions on a manifold M then the bracket is called a *Poisson bracket* on M, and the pair $(M, \{\ ,\ \})$ is called a *Poisson manifold*.

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A Lie bracket satisfies : $\{f, g\} = -\{g, f\}$ and $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$, for all $f, g, h \in C^\infty(M)$.

Poisson geometry

A Poisson structure on M can be equivalently defined by a bivector field $\pi \in \Gamma(\Lambda^2 TM)$ related to the Poisson bracket as follows:

$$\{f, g\} = \pi(df, dg),$$

for all $f, g, h \in C^\infty(M)$.

Remark:

- Poisson manifolds arise as phase spaces for classical mechanical systems. Poisson geometry has also applications in quantum mechanics and to noncommutative algebras.
- Derivations X_h of a Poisson algebra look like the inner derivations of a noncommutative algebra. In fact, there are strong analogies between Poisson geometry and noncommutative algebra.

Examples

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- Symplectic structures. A symplectic structure on a smooth even-dimensional manifold M is defined by a closed and non-degenerate 2-form ω on M.

  On a symplectic manifold M, any function h induces a unique vector field X_h called the hamiltonian vector field and defined by $\iota_{X_h}\omega = -dh$. Consequently, any symplectic structure gives rise to a Poisson bracket defined by :$\{f,g\} = \omega(X_f, X_g)$, $\forall\ f,g \in C^\infty(M)$.

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Symplectic structures=nondegenerate Poisson structures

On $M = \mathbb{R}^{2n}$ with the coordinates $(x_1 \cdots, x_n, y_1, \cdots, y_n)$ the standard symplectic form is $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.

Locally, any symplectic manifold looks like $(\mathbb{R}^{2n}, \omega_0)$.

Weinstein's splitting theorem

Theorem: Let $(M, \{\ ,\ \})$ be a Poisson manifold. Around any point $m_0 \in M$, there are local coordinates $(x_1 \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_{n-2k})$ such that

$$\{x_i, y_i\} = 1, \quad \{z_r, z_s\} = f_{rs}(z_1, \cdots, z_{n-2k}), \text{ with } f_{rs}(m_0) = 0,$$

$$\{x_i, y_j\} = 0, \ \{x_i, z_r\} = 0, \ \{y_i, z_r\} = 0, \ \forall\, 1 \leq i, j \leq k, 1 \leq r, s \leq n-2k.$$

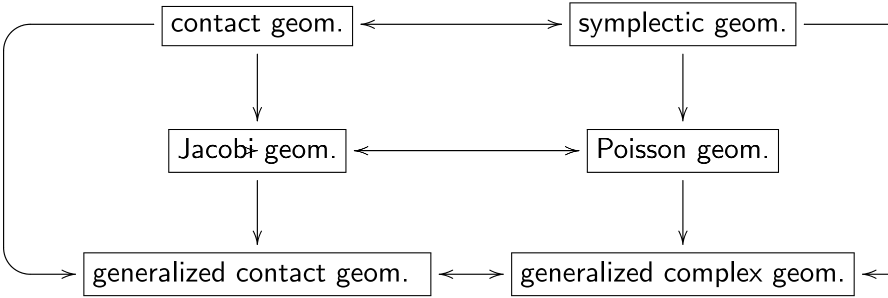
The Poisson structure is symplectic if $n = 2k$ for all $m_0 \in M$.

Remark:

This theorems says that a generic Poisson manifold is locally isomorphic to the product of an open subset of $\mathbb{R}^{2k}$ with the standard symplectic form and a Poisson manifold for which the Poisson tensor vanishes at some point m_0.

Inter-relations between geometries

The following diagram summarizes how various geometries are inter-related.



The big picture

Real Jacobi manifolds

- A (real) Jacobi structure on smooth manifold M is given by a "real" line bundle $L \to M$ and a Lie bracket $\{\cdot,\cdot\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ which is a bi-derivation, that is a derivation with respect to each entry.

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- By a derivation of L, we mean an $\mathbb{R}$-linear operation $\Delta : \Gamma(L) \to \Gamma(L)$ satisfying:

$$\Delta(fe) = f\Delta(e) + (\sigma(\Delta) \cdot f)e,$$

  where $\sigma(\Delta)$ is the symbol of Δ. Derivations of L can be identified with infinitesimal isomorphisms of L.

The gauge Lie algebroid and Jacobi brackets

Thus, derivations of L are sections of the vector bundle $DL \to M$ called the gauge (or Atiyah) Lie algebroid of L. The Lie bracket on $\Gamma(DL)$ is the commutator of derivations. Let J^1L be the first jet bundle of L, we have the vector bundle isomorphisms:

$$DL \simeq \mathrm{Hom}(J^1L, L) \quad \text{and} \quad J^1L \simeq \mathrm{Hom}(DL, L).$$

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Remark: Given a Jacobi manifold $(M, L, \{,\cdot,\cdot\})$, there is an associated 2-form

$$\mathcal{J} : \Gamma(\Lambda^2(J^1L)) \to \Gamma(L)$$

defined by:

$$\{\lambda, \mu\} = \mathcal{J}(j^1\lambda, j^1\mu),$$

Examples

- Given a contact structure $\xi \subseteq TM$ with its associated line bundle $L = TM/\xi$, the canonical projection $\Theta : TM \to TM/\xi$ induces a non-degenerate Jacobi tensor $\mathcal{J} : \Lambda^2 J^1 L \to L$. Conversely, every non-degenerate Jacobi structure on L corresponds to a contact structure on (M, L).

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- Any Poisson structure on M is determined by a bivector $\pi \in \Gamma(\Lambda^2 TM)$ called a Poisson tensor. Such a Poisson tensor corresponds to a Jacobi manifold on the trivial line bundle $L = M \times \mathbb{R}$. The first jet bundle of L is $J^1 L = T^*M \oplus \mathbb{R}$. Its associated 2-form $\mathcal{J} : \Lambda^2(J^1 L) \to L$ can be written in the matrix form: $\mathcal{J}_\pi = \begin{pmatrix} \pi & 0 \\ 0 & 0 \end{pmatrix}$, where π is the Poisson bi-vector field defining the Jacobi structure.

Real Jacobi vs Poisson structures

A Poisson tensor π on a smooth manifold $\widetilde{M}$ is homogeneous with respect to a vector field $Z \in \mathfrak{X}(M)$ if $\mathcal{L}_Z \pi = -\pi$.

Theorem: Jacobi structures on (M, L) are in one-to-one correspondence with homogeneous Poisson structures on $\widetilde{M} = L^* \smallsetminus \{0_M\}$ with respect to the Euler vector field on $\widetilde{M}$.

To prove this theorem, one uses the homogenization scheme.

What's the homogenization scheme?

- There is an equivalence between the categories of line bundles and principal $\mathbb{R}^\times$-bundles, where $\mathbb{R}^\times$ denotes the multiplicative group of non-zero reals.

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- Every section $\lambda \in \Gamma(L)$ can be identified with a fiber-wise linear function on L^*. By restriction, it can be considered as a homogeneous function de degree one on $\widetilde{M}$, denoted $\widetilde{\lambda}$. The correspondence $\lambda \mapsto \widetilde{\lambda}$ is one-to-one.

Complex manifolds

• An almost complex structure on a smooth real manifold M is a bundle map $J : TM \to TM$ such that $J^2 = -\mathrm{id}$.

• An almost complex structure J is called a complex structure if its Nijenhuis torsion

$$N_J : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is identically zero, i.e.

$$N_J(X, Y) = [JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) = 0, \ \forall X, Y \in \mathfrak{X}(M).$$

In this case (M, J) is called a complex manifold.

Complex manifolds

- The Newlander-Nirenberg theorem (1957) states that the condition $N_J = 0$ is equivalent to the existence of an holomorphic atlas for M.

- The complex tangent bundle splits as: $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}M$, where $T^{1,0}$ is the holomorphic tangent bundle and $T^{0,1}$ the an-tiholomorphic tangent bundle of M.

- In a holomorphic coordinate system $z_k = x_k + iy_k$, $k = 1, \cdots, n$, we have :

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad \text{and} \quad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.$$

Locally any n-dimensional complex manifolds looks like $\mathbb{C}^n$.

Complex contact manifolds

A complex contact manifold is a complex manifold (M, J) of dimension $\dim_{\mathbb{C}} M = 2n + 1$ together with an open covering $(\mathcal{U}_i)_{i \in I}$ by coordinate neighborhoods such that:

1. There is a holomorphic 1-form θ_i on each $\mathcal{U}_i$ such that

$$\theta_i \wedge (d\theta_i)^n \neq 0;$$

2. If $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ then there is a nowhere vanishing holomorphic function f_{ij} on $\mathcal{U}_i \cap \mathcal{U}_j$ such that

$$\theta_i = f_{ij} \theta_j.$$

• Complex contact manifolds are also called holomorphic contact manifolds.

Complex contact manifolds

- Any complex contact structure on M defines a holomorphic subbundle $\mathcal{H} \subseteq TM$ given by: $\mathcal{H} = \mathrm{Span}\{\ker(\theta_i), \quad i \in I\}$.
- The quotient $L = TM/\mathcal{H}$ is a holomorphic line bundle. The definition of a complex contact structure can be reformulated:

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Definition: A complex contact structure on a complex manifold M is given by a holomorphic vector subbundle $\mathcal{H} \subseteq T^{1,0}M$ of rank $2n$ which is completely non-integrable in the sense that the map:

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• We have : $0 \to \mathcal{H} \to T^{1,0}M \overset{\theta_{\mathcal{H}}}{\to} L \to 0$, where $\theta_{\mathcal{H}}$ is the canonical projection viewed as a hol. L-valued 1-form on X.

Important remark

Remark:

1. Let M be simply connected compact complex manifold. Any two complex contact structures on M are equivalent via some biholomorphic isomorphism of M.
2. The above property does not hold for real contact manifolds. For example, the 3-sphere $S^3 \subseteq \mathbb{R}^4$ carries various non-equivalent contact structures.

Example: the complex Heisenberg group $H_{\mathbb{C}}$

Example

The complex Heisenberg group $H_{\mathbb{C}}$ is a subgroup of $GL(3, \mathbb{C})$:

$$H_{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

The left invariant 1-form $\theta = \frac{1}{2}(dz_3 - z_2 dz_1)$ defines a complex contact structure on $H_{\mathbb{C}}$.

Example: complex projective spaces

• Given any compact complex manifold M, its projectivized tangent bundle $\mathbb{P}(T_M)$ is a contact manifold.

• Let G be a simple complex group and let $\mathbb{P}(\mathfrak{g})$ be the projectivized Lie algebra. The unique closed orbit for the adjoint action $\mathcal{O}$ of the Lie group G on $\mathbb{P}(\mathfrak{g})$ is a complex contact manifold.

Conjecture: The above two examples are the only contact projective manifolds (up to equivalence).

• This conjecture was proven to be true in dimension 3 by Ye.
• In dimension greater than 3, positive partial results were obtained by Demailly and Lebrun.

Enlarging the family of complex contact structures

• Since the existence of a complex contact structure on a manifold M imposes strong topological constraints on M, we want to weaken the condition in order to get a larger family of geometric objects. This leads to holomorphic Jacobi structures

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Definition: Let (M, J) be a complex manifold and let L be a holomorphic line bundle. A holomorphic Jacobi structure on (M, J) is defined by a Lie bracket on the sheaf Γ_L of holomorphic sections of L which is a first order bi-differential operator, in other words, one has a Lie bracket $\{\cdot, \cdot\} : \Gamma_L \times \Gamma_L \to \Gamma_L$ such that

$$\{\lambda_1, f\lambda_2\} = f\{\lambda_1, \lambda_2\} + (\sigma_{\lambda_1}(f))\lambda_2,$$

where f is a holomorphic function and σ_{λ_1} a holomorphic vector

Examples of holomorphic Jacobi manifolds

Examples of holomorphic Jacobi manifolds include

1. Holomorphic vector fields on complex manifolds
2. Holomorphic Poisson manifolds
3. Holomorphic contact structures
4. The dual $\mathfrak{g}^*$ of a Lie algebra endowed with a 1-cocycle.
5. Projective space $\mathbb{CP}(\mathfrak{g}^*)$ of the dual of a complex Lie algebra,

Holomorphic Jacobi manifolds

Similarly to the real case, we have the following result:

Theorem: Let L be a holomorphic line bundle on a complex manifold M. Holomorphic Jacobi stuctures on (M, L) are in one-to-one correspondence with holomorphic homogeneous Poisson brackets on $\widetilde{M} = L^* \setminus \{0_M\}$.

Thus, holomorphic Jacobi manifolds are richer versions of holomorphic Poisson manifolds.

Holomorphic Jacobi manifolds

To summarize, we have:

Jacobi manifolds	$\Longleftarrow$	holomorphic Jacobi	$\Longrightarrow$	generalized contact
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
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THANK YOU